ISRAEL JOURNAL OF MATHEMATICS 94 (1996), 125–146

ON CRITICAL FUJITA EXPONENTS FOR HEAT EQUATIONS WITH NONLINEAR FLUX CONDITIONS ON THE BOUNDARY*

BY

VICTOR A. GALAKTIONOV**

Keldysh Institute of Applied Mathematics
Russian Acad. Sci., Miusskaya Sq., 4, 125047 Moscow, Russia
e-mail: vgalak@ccuam3.sdi.uam.es

AND

HOWARD A. LEVINE

Department of Mathematics
Iowa State University, Ames, Iowa 50011, U.S.A.
e-mail: halevine@iastate.edu

ABSTRACT

We consider nonnegative solutions of initial-boundary value problems for parabolic equations $u_t = u_{xx}$, $u_t = (u^m)_{xx}$ and $u_t = (|u_x|^{m-1}u_x)_x$ $(m > 1)$ for $x > 0$, $t > 0$ with nonlinear boundary conditions $-u_x = u^p$, $-(u^m)_x = u^p$ and $-|u_x|^{m-1}u_x = u^p$ for $x = 0$, $t > 0$, where $p > 0$. The initial function is assumed to be bounded, smooth and to have, in the latter two cases, compact support. We prove that for each problem there exist positive critical values p_0, p_c (with $p_0 < p_c$) such that for $p \in (0, p_0]$, all solutions are global while for $p \in (p_0, p_c]$ any solution $u \not\equiv 0$ blows up in a finite time and for $p > p_c$ small data solutions exist globally in time while large data solutions are nonglobal. We have $p_c = 2$, $p_c = m + 1$ and $p_c = 2m$ for each problem, while $p_0 = 1$, $p_0 = \frac{1}{2}(m + 1)$ and $p_0 = 2m/(m + 1)$ respectively.

---

* This work was done during visits of the first author to Iowa State University and the Institute for Mathematics and its Applications at the University of Minnesota. The second author was supported in part by NSF Grant DMS-9102210.

** *Current address:* Departamento de Matematicas, Universidad Autonoma de Madrid, 28049 Madrid, Spain.

Received March 17, 1994 and in revised form October 13, 1994

1. Main results

The problem of determining critical Fujita exponent is an interesting one in the general theory of blowing-up solutions to different nonlinear evolution equations of mathematical physics. See the survey [L2] where a full list of references is given.

In this paper we consider three problems for parabolic equations of nonlinear heat conduction type with nonlinear boundary constraints having critical Fujita exponents.

The first one is formulated for the heat equation:

$$\text{(H)}\qquad \begin{aligned} u_t &= u_{xx} && \text{for } x>0, \quad t>0;\\ -u_x &= u^p && \text{for } x=0, \quad t>0;\\ u(0,x) &= u_0(x)\geq 0 && \text{for } x>0; \quad \sup u_0<\infty,\\ -u_0'(0) &= u_0^p(0), \end{aligned}$$

where p is fixed and satisfies

$$p > p_0 = 1. \tag{1.1}$$

The second problem is a generalization of the previous one to the porous media equation:

$$\text{(P}_m\text{)}\qquad \begin{aligned} u_t &= (u^m)_{xx} && \text{for } x>0, \quad t>0;\\ -(u^m)_x &= u^p && \text{for } x=0, \quad t>0;\\ u(0,x) &= u_0(x)\geq 0 && \text{for } x>0; \quad \sup u_0<\infty, \end{aligned}$$

$$\begin{aligned} &\sup|(u_0^{m-1})'|<\infty, \ u_0 \text{ has compact support},\\ &-(u_0^m)'(0) = u_0^p(0). \end{aligned}$$

Here $m>1$ and

$$p > p_0 = \frac{m+1}{2}. \tag{1.2}$$

The third problem is stated for the heat conduction equation with gradient dependent diffusion (again with $m>1$):

$$\text{(G}_m\text{)}\qquad \begin{aligned} u_t &= (|u_x|^{m-1}u_x)_x && \text{for } x>0, \quad t>0;\\ -|u_x|^{m-1}u_x &= u^p && \text{for } x=0, \quad t\geq 0;\\ u(0,x) &= u_0(x)\geq 0 && \text{for } x>0; \quad \sup u_0<\infty, \end{aligned}$$

$$\sup |u_0'| < \infty, \ u_0 \text{ is compactly supported,}$$
$$-|u_0'(0)| = u_0^p(0).$$

The constant p is assumed to satisfy the assumption

$$p > p_0 = \frac{2m}{m+1}. \tag{1.3}$$

Local in time existence of positive classical solutions of the problem (H) (see [F]) or nonnegative weak compactly supported solutions to problems (P_m) and (G_m) and comparison arguments are well-known. These questions for the degenerate equations given in (P_m) and (G_m) are discussed in a survey [K]; see the full list of references therein. Blowing-up solutions of the problem (H) have been studied in many papers, see, e.g., [LP], [BGK], [Fi], [FQ], [GKS], [L2], [SGKM, Chapter III]. Notice that lower bounds (1.1), (1.2) and (1.3) for problems (H), (P_m) and (G_m) resp. are needed for blowing-up of any solution with large enough initial data. It is easy to show that if $p \leq p_0$ ($p > 0$) then for arbitrary initial functions the solution is global in time. This is done in Remarks 2.2, 3.1 and 4.1.

We denote by p_c ($> p_0$) the critical exponent of Fujita type. By definition, this means that p_c has the following properties:

(i) If $p \leq p_c$ ($p > p_0$), then $u(x,t) \not\equiv 0$ blows-up in a finite time for all nontrivial u_0,

(ii) if $p > p_c$, then $u(x,t)$ is global in time for "small" $u_0 \not\equiv 0$.

Case (i) is called the blow up case while case (ii) is called the global existence case. The terminology used in the second case is employed because there are global, nontrivial solutions in this case which is not the case in case (i). Analogously, we could call p_0 the critical global existence exponent since it has the following property: If $p > p_0$, there are always nonglobal solutions of each of the three problems listed above while if $p \leq p_0$, every solution of each the above three problems is global.

We now state the main results of the paper.

THEOREM H: *For problem* (H) $p_c = 2$, $p_0 = 1$.

THEOREM P: *For problem* (P_m) $p_c = m+1$, $p_0 = \frac{1}{2}(m+1)$.

THEOREM G: *For problem* (G_m) $p_c = 2m$, $p_0 = \frac{2m}{m+1}$.

It is interesting to compare results given above with the critical value (p_c^*) for the Cauchy problem for equations with source and the same heat operator.

Consider first the equation corresponding to Theorem H and P:

$$u_t = (u^m)_{xx} + u^p \qquad \text{for } x \in \mathbb{R}, \quad t > 0, \tag{1.4}$$

where $m \geq 1$ and $p > 1$. The critical exponent for (1.4) is $p_c^* = m + 2$, see [GKMS] and [L2], [SGKM, p. 208]. For the equation with gradient-like diffusion and source

$$u_t = (|u_x|^{m-1} u_x)_x + u^p \qquad \text{for } x \in \mathbb{R}, \quad t > 0 \tag{1.5}$$

the critical value is $p_c^* = 2m+1$, see [G1] and [L2]. The fact that in both of these last two equations, p_c^* belongs to the blow up case has been proved in [G2].

Thus in both cases we have $p_c = p_c^* - 1$.

2. Proof of Theorem H

We shall use a modification of an argument of Kaplan introduced in [BL].

2.1 *$u(x,t)$ blows up for large u_0.*

Let

$$\phi(x) = 2\sqrt{\frac{k}{\pi}} e^{-kx^2} \quad \text{in } \mathbb{R}_+, \tag{2.1}$$

where $k > 0$ is a constant. Then one can see $\phi'(0) = 0$,

$$\phi''(x) \geq -2k\phi(x) \quad \text{in } \mathbb{R}_+, \tag{2.2}$$

and

$$\int_0^\infty \phi(x)dx = 1. \tag{2.3}$$

Define

$$F(t) = \int_0^\infty u(x,t)\phi(x)dx. \tag{2.4}$$

Assume now that $u_0(x)$ is a non increasing function and $u_0'(x) \rangle 0$ as $x \rangle \infty$. Then by the Maximum Principle [F, Chapter II] we have that $u_x(x,t) \leq 0$ for $x \in \mathbb{R}_+$ and any t in the existence interval. Hence

$$F(t) \leq u(0,t) \quad \text{for } t \geq 0. \tag{2.5}$$

Then by multiplying the heat equation by $\phi(x)$ and integrating over $(0,\infty)$ and using the fact that under the above hypotheses, $u_x(x,t)\rangle 0$ as $x\rangle\infty$, we conclude that $F(t)$ satisfies

$$F'(t) = \int_0^\infty \phi u_{xx}\,dx = -\phi(0)u_x(0,t) + \int_0^\infty u\phi_{xx}\,dx.$$

Hence, it follows from (2.2) and (2.5) that

$$\begin{aligned} F'(t) &\geq \phi(0)F^p(t) - 2kF(t) \\ &= 2\sqrt{\frac{k}{\pi}}F(t)\left[F^{p-1}(t) - \sqrt{k\pi}\right] \end{aligned} \tag{2.6}$$

for $t > 0$. Thus, (2.6) implies that if

$$F(0) = \int_0^\infty u_0(x)\phi(x)\,dx > (\sqrt{k\pi})^{1/(p-1)}, \tag{2.7}$$

then $F(t)$ blows up in a finite time. Hence, by (2.5) $u(0,t)$ is not bounded on this finite interval. The analysis for arbitrary u_0 large near $x = 0$ and not necessarily monotone, can be easily done by comparison.

Remark 2.1: From the concavity arguments of [LP] we see that for $p > 1$ the solution blows up in a finite time whenever the initial function $u_0(x)$ satisfies

$$\frac{1}{2}\int_0^\infty |u_0'|^2\,dx < \frac{1}{p+1}u_0^{p+1}(0).$$

This is simply the statement that if the initial potential energy is negative, the solution cannot be global. A similar statement can be written down and established using concavity arguments for problems (P_m) and (G_m). However we shall proceed somewhat differently in those cases.

2.2 *Any $u \not\equiv 0$ blows up if $1 < p < 2$.*

Consider (2.7) more carefully. It is equivalent to the inequality

$$2\sqrt{\frac{k}{\pi}}\int_0^\infty u_0(x)e^{-kx^2}\,dx > (\sqrt{k\pi})^{1/(p-1)}. \tag{2.8}$$

Since $u_0 \not\equiv 0$, we conclude that (2.8) is valid for arbitrarily small $k > 0$ if

$$\frac{1}{2} < \frac{1}{2(p-1)},$$

i.e., $p < 2$ whence the result.

2.3 *Global small solutions for $p > 2$.*

We shall look for a global supersolution of the following self-similar type:

$$\bar{u}(x,t) = (T+t)^{-1/(2(p-1))} f(\eta), \quad \eta = \frac{x}{(T+t)^{1/2}}, \tag{2.9}$$

where $T > 0$ is a constant. By substituting (2.9) into the problem (H) we deduce that $\bar{u}(x,t)$ is a supersolution if the function $f \geq 0$ satisfies

$$f''(\eta) + \frac{\eta}{2} f'(\eta) + \frac{1}{2(p-1)} f(\eta) \leq 0 \quad \text{for } \eta > 0, \tag{2.10}$$

$$-f'(0) \geq f^p(0). \tag{2.11}$$

Set

$$f(\eta) = Ae^{-\alpha(\eta+b)^2}, \tag{2.12}$$

where A, α, b are positive constants. Then (2.10) becomes

$$\left[\frac{1}{2(p-1)} - 2\alpha + 4\alpha^2 b^2\right] + \alpha b(8\alpha - 1)\eta + \alpha(4\alpha - 1)\eta^2 \leq 0 \tag{2.13}$$

for $\eta > 0$, and (2.11) is valid if

$$2\alpha b e^{\alpha b^2(p-1)} \geq A^{p-1}. \tag{2.14}$$

One can see that (2.13) is valid for any $0 < b << 1$ if

$$\frac{1}{2(p-1)} - 2\alpha < 0 \quad \text{and} \quad 4\alpha - 1 < 0, \tag{2.15}$$

i.e. if $p > 2$. Then (2.14) holds if A is small enough. Thus for $p > 2$ we have constructed a class of global supersolutions of the form given in (2.9). Therefore if $u_0(x) \leq \overline{u}(x,0)$ for $x > 0$, then $u \leq \overline{u}$ on $\mathbb{R}_+ \times [0,T)$ where T is the length of the existence interval for u by comparison. To see this, we first note that the representation formula,

$$u(x,t) = \int_0^\infty G(x,y,t)u_0(y)dy + \int_0^t u^p(0,\eta)G(x,0,t-\eta)d\eta,$$

where

$$G(x,y,t) = (4\pi t)^{-1/2}\left[e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}\right]$$

is the Green's function for the heat operator in half-space satisfying $G_x(x,0,t) \equiv 0$, and the contraction mapping principle can be used to establish the existence of a local, in time, solution with the same initial data as the supersolution. Since the supersolution is global, the local solution can be extended to a global solution. The details are more or less standard and we therefore omit them.

Remark 2.2: If $p \leq 1$ all solutions with bounded initial values are global. In order to establish this claim we define, for any $C, \alpha > 0$,

$$\overline{u}(x,t) = e^{\alpha^2 t}(e^{-\alpha x} + C)$$

in the closed quarter plane $\{x \geq 0,\ t \geq 0\}$. If we fix $\alpha > 1$ and take

$$\alpha = (1+C)^p,$$

then it is easy to see that $\overline{u}$ will be a supersolution for problem H with $\overline{u}(x,0) \geq C$ which we may make as large as we please by choosing C sufficiently large. (The condition on p is used in order to insure that the inequality $e^{(1-p)\alpha^2 t} \geq 1$ holds for $t \geq 0$.)

2.4 *Any $u \not\equiv 0$ blows up for $p = 2$.*

We now use a somewhat different version of Kaplan's argument introduced in [BL]. From the representation formula preceding Remark 2.2, we have, with $p = 2$,

$$u(x,t) = \int_0^\infty G(x,y,t)u_0(y)dy + \int_0^t u^2(0,\eta)G(x,0,t-\eta)d\eta. \tag{2.16}$$

Thus, we have

$$u(0,t) \geq C_0 \int_0^t (t-\eta)^{-\frac{1}{2}} u^2(0,\eta)d\eta \geq C_0 t^{-\frac{1}{2}} \int_0^t u^2(0,\eta)\, d\eta \equiv C_0 t^{-\frac{1}{2}} F(t), \tag{2.17}$$

where $C_0 = \sqrt{\frac{2}{\pi}}$. Then $\sqrt{tF'(t)} \geq C_0 F(t)$ or

$$\frac{F'}{F^2} \geq C_0^2 t^{-1}. \tag{2.18}$$

After a quadrature, we have, for $t \geq t_0 > 0$,

$$\frac{1}{F(t_0)} \geq -\frac{1}{F(t)} + \frac{1}{F(t_0)} \geq C_0^2 \ln \frac{t}{t_0}.$$

Thus, $F(t)$ blows up in finite time and by (2.17) the same is true for $u(0,t)$.

This completes the proof of Theorem H.

3. Proof of Theorem P

In this section we consider problem (P_m) with $m > 1$ and with p satisfying (1.2).

3.1 *$u(x,t)$ blows up for large u_0.*

We begin with construction of a nonglobal subsolution of the self-similar form

$$\underline{u}(x,t) = (T-t)^{-\frac{1}{2p-(m+1)}}\theta(\eta), \qquad \eta = \frac{x}{(T-t)^{\frac{p-m}{2p-(m+1)}}}, \tag{3.1}$$

where $T > 0$ is a given constant. One can see that (3.1) is a subsolution if the function $\theta \geq 0$ is such that

$$(\theta^m)''(\eta) - \frac{p-m}{2p-(m+1)}\eta\theta'(\eta) - \frac{1}{2p-(m+1)}\theta(\eta) \geq 0 \tag{3.2}$$

for $\eta \in \{\eta > 0 \,|\, \theta(\eta) > 0\}$ and

$$-(\theta^m)'(0) \leq \theta^p(0). \tag{3.3}$$

We claim that (3.2), (3.3) admits a solution of the form

$$\theta(\eta) = A(a-\eta)_+^{\frac{1}{m-1}} \tag{3.4}$$

for positive constants A, a. To see this we first observe that for some $A, a > 0$, (3.2) holds with such $\theta(\cdot)$ if

$$\frac{m}{(m-1)^2}A^{m-1} \geq \frac{a}{2p-(m+1)}. \tag{3.5.1}$$

In order to verify this, we see after substitution that as long as $\eta < a$, (3.2) will hold if

$$\frac{mA^{m-1}}{(m-1)^2} + \frac{(p-1)\eta}{(m-1)(2p-(m+1))} - \frac{a}{2p-(m+1)} \geq 0.$$

Likewise we claim that (3.3) will hold for such $\theta(\cdot)$ provided

$$A^{p-m}a^{\frac{p-1}{m-1}} \geq \frac{m}{m-1} \tag{3.5.2}$$

for the same constants A, a. This is clear. If we now choose $a = cA^{m-1}$ where $c \leq m(2p-m-1)/(m-1)^2$, and then choose A sufficiently large, we see that both (3.5.1) and (3.5.2) hold. Notice that with the choice of θ in (3.4) the function (3.1) is a weak subsolution of the problem considered, see [K]. Thus, if for any given T and A, a satisfying (3.5) the initial function u_0 is such that

$$u_0(x) \geq \underline{u}(x,0) \quad \text{for } x > 0, \tag{3.6}$$

then $u(x,t) \geq \underline{u}(x,t)$ in $\mathbb{R}_+ \times (0,T)$, and hence $u(x,t)$ blows up in a finite time which is not larger than T.

3.2 *Any $u \not\equiv 0$ blows up* if $p_0 < p < m+1$.

We now use the idea used for a different problem in [GKMS], see also [SGKM, p. 208].

We first notice that problem (P_m) admits the following well-known self-similar solution (the so-called Zel'dovich–Kompaneetz–Barenblatt profile [K], [SGKM], Chapter I):

$$u_B(x,t) = (\tau+t)^{-\frac{1}{m+1}} g(\zeta), \quad \zeta = \frac{x}{(\tau+t)^{\frac{1}{m+1}}}, \tag{3.7}$$

where $\tau \geq 0$ is an arbitrary constant and $g \geq 0$ satisfies

$$(g^m)''(\zeta) + \frac{\zeta}{m+1} g'(\zeta) + \frac{1}{m+1} g(\zeta) = 0, \tag{3.8}$$

$$g'(0) = 0, \tag{3.9}$$

and hence

$$g(\zeta) = C_m (c^2 - \zeta^2)_+^{\frac{1}{m-1}}, \tag{3.10}$$

where $c > 0$ is an arbitrary constant and

$$C_m = \left[\frac{m-1}{2m(m+1)}\right]^{\frac{1}{m-1}}.$$

The condition (3.9) implies that $u_B(x,t)$ is a subsolution to problem (P_m).

By using well-known properties of weak solutions of problem (P_m) ([K]) we deduce that there exists $t_0 \geq 0$ such that

$$u(0, t_0) > 0. \tag{3.11}$$

Since $u(x, t_0)$ is a continuous function, there exist $\tau > 0$ large enough and small $c > 0$ such that

$$u(x, t_0) \geq u_B(x, t_0) \quad \text{for } x > 0. \tag{3.12}$$

Then by comparison we deduce that

$$u(x,t) \geq u_B(x,t) \quad \text{for } x > 0,\ t > t_0. \tag{3.13}$$

We now prove that there exist $t_* \geq t_0$ and T large enough so that

$$u_B(x,t_*) \geq \underline{u}(x,0) \quad \text{for } x > 0, \tag{3.14}$$

where $\underline{u}(x,t)$ is the subsolution given by (3.1) and (3.4). By using the space-time structure of both functions $u_B(x,t_*)$ and $\underline{u}(x,0)$ given in (3.7) and (3.1) respectively , we conclude that (3.14) is valid if

$$(\tau + t_*)^{-\frac{1}{m+1}} >> T^{-\frac{1}{2p-(m+1)}} \tag{3.15}$$

and

$$(\tau + t_*)^{\frac{1}{m+1}} >> T^{\frac{p-m}{2p-(m+1)}}. \tag{3.16}$$

One can see from (3.15), (3.16) that such t_* and T large enough exist if

$$T^{\frac{1}{2p-(m+1)}} >> T^{\frac{p-m}{2p-(m+1)}} \tag{3.17}$$

for arbitrarily large T. This implies that

$$\frac{1}{2p-(m+1)} > \frac{p-m}{2p-(m+1)}, \tag{3.18}$$

i.e., $p < m+1$. Hence, if $p \in (p_0, m+1)$, every nontrivial, nonnegative solution of (P_m) blows up in finite time.

3.3 *Global small solutions for $p > m+1$.*

We shall seek a global supersolution of the self-similar form

$$\bar{u}(x,t) = (T+t)^{-\frac{1}{2p-(m+1)}} f(\eta), \quad \eta = \frac{x}{(T+t)^{\frac{p-m}{2p-(m+1)}}} \tag{3.19}$$

where the function $f(\eta) \geq 0$ is such that (cf. (3.2), (3.3))

$$(f^m)''(\eta) + \frac{p-m}{2p-(m+1)}\eta f'(\eta) + \frac{1}{2p-(m+1)} f(\eta) \leq 0 \tag{3.20}$$

for $\eta \in \{\eta > 0 \,|\, f(\eta) > 0\}$ and

$$-(f^m)'(0) \geq f^p(0). \tag{3.21}$$

It is easily seen that (3.20), (3.21) does not admit a solution of the form (3.4). Therefore we shall look for a solution which is a translate of that given by (3.10):

$$f(\eta) = Bg(\eta + b), \tag{3.22}$$

where $B > 0$ and $b \in (0, c)$ are constants. It follows from results given in [K] that the function (3.19), (3.22) is regular enough to be a supersolution. By using the fact that $g(\eta + b)$ satisfies the identity

$$(g^m)''(\eta+b) \equiv -\frac{1}{m+1}(\eta+b)g'(\eta+b) - \frac{1}{m+1}g(\eta+b), \quad 0 < \eta < c-b, \tag{3.23}$$

we conclude that function (3.22) is the solution of the inequality (3.20) if

$$\begin{aligned} &-g'(\eta+b)\left[\frac{B^{m-1}}{m+1}(\eta+b) - \frac{p-m}{2p-(m+1)}\eta\right] \\ &+g(\eta+b)\left[\frac{1}{2p-(m+1)} - \frac{B^{m-1}}{m+1}\right] \leq 0 \end{aligned} \tag{3.24}$$

for $\eta \in (0, c-b)$. By substituting the function $g(\eta+b)$ given by (3.10) into (3.24) we arrive at the inequality for $z = \eta + b \in (b, c)$:

$$\begin{aligned} -\left(\frac{1-B^{m-1}}{m-1}\right) z^2 + \left(\frac{2}{m-1}\,\frac{p-m}{2p-(m+1)}\right) bz& \\ - c^2\left(\frac{B^{m-1}}{m+1} - \frac{1}{2p-(m+1)}\right) &\leq 0. \end{aligned} \tag{3.25}$$

Since $p > m+1$ we can choose $B > 0$ such that

$$\frac{m+1}{2p-(m+1)} < B^{m-1} < 1.$$

Now set

$$\begin{aligned} e_1 &\equiv \frac{1-B^{m-1}}{m-1}, \\ e_2 &\equiv \frac{2(p-m)}{(m-1)(2p-(m+1))}, \\ e_3 &\equiv \frac{1}{m+1}\left[B^{m-1} - \frac{m+1}{2p-(m+1)}\right], \end{aligned}$$

and note that all of the $e_i > 0$. Setting

$$R(z) \equiv -e_1 z^2 + e_2 bz - c^2 e_3, \quad z_* = \frac{e_2 b}{2e_1},$$

we see that for all $z \in \mathbb{R}$

$$R(z) \le R(z_*) \equiv b^2 \left(\frac{e_2^2}{4e_1} - \alpha^2 e_3 \right) \le 0,$$

where we have set $c = \alpha b$ for $\alpha \equiv \max(1, \frac{e_2}{2\sqrt{e_1 e_3}})$.

Finally, we note that for the function defined in (3.22) inequality (3.21) is equivalent to

$$-B^m m g^{m-1}(b) g'(b) \ge B^p g^p(b).$$

Or (see (3.10)),

$$(BC_m)^{p-m}(c^2 - b^2)^{\frac{p-1}{m-1}} \le \frac{2bm}{m-1} \tag{3.26}$$

or, equivalently,

$$(BC_m)^{p-m}(\alpha^2 - 1)^{\frac{p-1}{m-1}} b^{\frac{2p-(m+1)}{m-1}} \le \frac{2m}{m-1}, \tag{3.27}$$

which is valid if $b > 0$ is small enough and α is as above.

Thus, for $p > m + 1$ there exists a nontrivial global supersolution, and hence a class of small global solutions satisfying $u \le \overline{u}$ in $\mathbb{R}_+ \times \mathbb{R}_+$.

Remark 3.1: The fact that if $p \in \left[0, \frac{m+1}{2}\right]$ then the solution is global in time, can be easily proved by comparison with a global self-similar solution for $p = \frac{m+1}{2} \equiv p_0$:

$$u_*(x,t) = e^{\alpha(T+t)} h(\xi), \qquad \xi = \frac{x}{e^{\beta(T+t)}}, \tag{3.28}$$

where $T > 0$, $\alpha > 0$ are constants and $\beta = \frac{\alpha(m-1)}{2}$. The function $h(\xi) \ge 0$ satisfies

$$(h^m)''(\xi) + \frac{\alpha(m-1)}{2} \xi h'(\xi) - \alpha h(\xi) = 0 \qquad \text{for } \xi > 0, \tag{3.29}$$

$$-(h^m)'(0) = h^{p_0}(0). \tag{3.30}$$

There exists a unique compactly supported solution $h \not\equiv 0$ to the problem (3.29), (3.30), see [GP] and references therein. Therefore, for the case $p = p_0$ we can choose $T > 0$ large enough such that $u_0(x) \le u_*(x,0)$ for $x > 0$ and hence by comparison $u \le u_*$ in $\mathbb{R}_+ \times \mathbb{R}_+$, i.e., $u(x,t)$ is global. If $p < p_0$, then $u_*(x,t)$ is a global supersolution whenever $u_*(0,t) \ge 1$. Hence, we can also argue by comparison.

3.4 *Any $u \not\equiv 0$ blows up if $p = m + 1$.*

Here we argue as in [G2]. Assume that for $p = m+1$ there exists a global (in time) nonnegative solution $u \not\equiv 0$ to problem (P_m). Without loss of generality, we may suppose that $u_0(0) > 0$ and therefore

$$u_0(x) \geq g(x+b) \equiv C_m \left(c^2 - (x+b)^2\right)_+^{\frac{1}{m-1}} \tag{3.31}$$

on $\mathbb{R}_+$ provided that $c > 0$ and $b \in (0, c)$ are sufficiently small. It is convenient to introduce the rescaled function which corresponds to the space-time structure of the self-similar solution given in (3.7):

$$\theta(\zeta, \tau) = (1+t)^{\frac{1}{m+1}} u(\zeta(1+t)^{\frac{1}{m+1}}, t), \tag{3.32}$$

where $\tau \equiv \ln(1+t)$ denotes the new time. Then the function θ is a global solution of the problem

$$\theta_\tau = \mathfrak{A}(\theta) \equiv (\theta^m)_{\zeta\zeta} + \frac{1}{m+1}(\zeta\theta)_\zeta \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \tag{3.33}$$

$$-(\theta^m)_\zeta = \theta^{m+1} \qquad \text{for } \zeta = 0,\ \tau > 0, \tag{3.34}$$

$$\theta(\zeta, 0) = u_0(\zeta) \qquad \text{for } \zeta > 0. \tag{3.35}$$

Denote by $\underline{\theta}(\zeta,\tau)$ the solution of the problem (3.33)–(3.35) with initial data $g(x+b)$ given in (3.31). It follows from (3.31) that

$$\theta(\zeta,\tau) \geq \underline{\theta}(\zeta,\tau) \tag{3.36}$$

on $\mathbb{R}_+ \times \mathbb{R}_+$. Therefore $\underline{\theta}$ is also a global solution.

We now discuss some important properties of $\underline{\theta}(\zeta,\tau)$. Observe first that since $\underline{\theta}_\zeta$ is initially nonpositive and $(\underline{\theta}^m)_\zeta$ is nonpositive on the boundary, it follows that $\underline{\theta}(\zeta,\tau)$ is nonincreasing in ζ.

Proposition 3.1: *Let*

$$c^2 = b^2 + \left[\frac{2mb}{(m-1)C_m}\right]^{\frac{m-1}{m}}. \tag{3.37}$$

Then $\underline{\theta}(\zeta,\tau)$ is nondecreasing in τ on $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof: First, we see that (3.37) implies the compatibility condition at the origin, that is (3.34) also holds when $\tau = 0$. Now set $z = \underline{\theta}_\tau$. Then by regularity [K] we have from (3.33), (3.34) that z satisfies

$$z_\tau = (m\underline{\theta}^{m-1} z)_{\zeta\zeta} + \frac{1}{m+1}(\zeta z)_\zeta \tag{3.38}$$

in the set $\{\underline{\theta} > 0\}$ and the boundary condition

$$-(m\underline{\theta}^{m-1}z)_\zeta = (m+1)\underline{\theta}^m z \tag{3.39}$$

in the set $\{\zeta = 0, \tau > 0\}$. (Notice that $\underline{\theta}(0,\tau) > 0$ by construction so that $\underline{\theta}$ is smooth on the boundary $\zeta = 0$ [K].)

Using identity (3.23) we obtain that the initial values for z satisfy

$$\begin{aligned} z(\zeta,0) &= \mathfrak{A}(g(\zeta+b)) \\ &= (g^m((\zeta+b))_{\zeta\zeta} + \frac{1}{m+1}(\zeta g(\zeta+b))_\zeta \\ &= -\frac{1}{m+1} b g_\zeta(\zeta+b) \geq 0 \end{aligned} \tag{3.40}$$

in $\{g(\zeta+b) > 0\}$.

Therefore the proposition will follow from (3.38)–(3.40) and the Maximum Principle if we can show that a weak solution $\underline{\theta}$ is the limit as $\epsilon \to 0$ of a monotone sequence $\{\underline{\theta}_\epsilon\}$ of strictly positive classical solutions having the same property as solutions of problem (3.33)–(3.35) with positive initial data $\underline{\theta}_\epsilon(\zeta,0) = \max(\epsilon, g(\zeta+b))$. (This is a rather standard regularization procedure [K].) In order to avoid the extra regularity assumption $\underline{\theta}_\epsilon \in C^{4,2}_{\zeta\tau}(Q) \cap C^{2,1}_{\zeta\tau}(\overline{Q})$ $(Q = \mathbb{R}_+ \times \mathbb{R}_+)$ which is necessary to justify the differentiation with respect to time, we use the finite difference approximation of the time derivative

$$(\underline{\theta}_\epsilon)_h \equiv \frac{\underline{\theta}_\epsilon(\zeta,\tau+h) - \underline{\theta}_\epsilon(\zeta,\tau)}{h}, \quad h > 0$$

(see [S]). That is, we write a linear parabolic problem for $(\underline{\theta}_\epsilon)_h$ instead of $(\underline{\theta}_\epsilon)_\tau$. The conclusion then follows by first passing to the limit as $h \to 0$ and then as $\epsilon \to 0$ (cf. [SGKM, Chapter V]).

PROPOSITION 3.2: *For any* $\zeta > 0$

$$+\infty > \lim_{\tau\to+\infty} \underline{\theta}(\zeta,\tau) = F(\zeta) \not\equiv 0. \tag{3.41}$$

Proof: We argue by contradiction. Assume that (3.41) fails for some $\zeta_0 > 0$ and hence that $\lim_{\tau\to+\infty} \underline{\theta}(\zeta_0,\tau) = +\infty$. Since $\underline{\theta}$ is non increasing in ζ, we conclude that

$$\lim_{\tau\to+\infty} \underline{\theta}(\zeta,\tau) = +\infty \tag{3.42}$$

uniformly on $[0, \zeta_0]$. However, this contradicts the estimates of subsection 3.1 as follows: From (3.42) we conclude that after a finite time $\tau_0 = e^{t_0} - 1$, the profile $\underline{\theta}(\zeta, \tau_0)$ in the original variables will satisfy (3.6) with suitable choices of T, A, a of the subsolution (3.1), (3.4). This implies that $\underline{\theta}(\zeta, \tau)$, which was assumed to be global, will blow up in a finite time which is not larger than $\ln(1 + t_0 + T)$.

PROPOSITION 3.3: *Under the above hypotheses, the function $F(\cdot)$ in (3.41) is a weak stationary solution of (3.33). That is,*

$$\mathfrak{A}(F) \equiv (F^m)_{\zeta\zeta} + \frac{1}{m+1}(\zeta F)_\zeta = 0 \tag{3.43}$$

for $\zeta > 0$.

Proof: This is an immediate consequence of the fact that the problem for $\underline{\theta}$ admits a Lyapunov function

$$\mathfrak{E}(\tau) \equiv \int_\delta^{\frac{1}{\delta}} [F(\zeta) - \underline{\theta}(\zeta, \tau)]\, d\zeta \geq 0,$$

where $\delta > 0$ is small and fixed. From Proposition 3.1 we see that $\mathfrak{E}(\cdot)$ is non-increasing and $\lim_{\tau\to\infty} \mathfrak{E}(\tau) = 0$. From regularity and monotonicity we also conclude that

$$\mathfrak{E}(\tau) = \int_\tau^\infty \int_\delta^{\frac{1}{\delta}} |\underline{\theta}_\tau(\zeta, s)|\, d\zeta\, ds$$

and hence that

$$\int_1^\infty \|\underline{\theta}(\cdot, s)\|_{L^1(\delta, \frac{1}{\delta})}\, ds < \infty.$$

In view of the known regularity of bounded solutions of the porous medium equation [K], this estimate and Proposition 3.1 permit us to pass to the limit as $\tau \to \infty$ in (3.33) and hence establish (3.43). See also [LaP, G2] for a similar analysis.

In order to finish the blow up argument in this case ($p = m + 1$) we observe that the only self-similar solutions of (3.43) are of the form given in (3.10). (This follows by uniqueness and a symmetrization argument.) In particular, $F(0)$ (> 0) is finite. Then, by passing to the limit in the boundary condition (3.34) for $\underline{\theta}$, noting Proposition 3.1, and the regularity of $F(\cdot)$ in the region where $F > 0$ [K], we conclude that

$$-(F^m)_\zeta = F^{m+1} \quad \text{for } \zeta = 0.$$

However, functions of the form (3.10) with $c > 0$ do not satisfy this boundary condition. Thus such a bounded, stationary solution does not exist and hence the stabilization (3.41) cannot hold. Therefore a global solution with initial values satisfying (3.31) is not possible. This completes the proof in the critical case.

4. Proof of Theorem G

We shall follow the same format as in the proof of the preceding two theorems. The constructions closely parallel (for a very good reason given in Remark 4.2 below) those of the preceding section and consequently the presentation will be somewhat terse.

4.1 *$u(x,t)$ blows up for large u_0.*

We construct a self-similar weak subsolution of the problem (G_m) of the form

$$\underline{u}(x,t) = (T-t)^{-\frac{m}{p(m+1)-2m}}\theta(\eta), \quad \eta = \frac{x}{(T-t)^{\frac{p-m}{p(m+1)-2m}}}, \tag{4.1}$$

where $T > 0$ is a given constant which blows up as $t \to T^-$. Assuming that $\theta \geq 0$ is non increasing and sufficiently smooth, then it must satisfy the inequalities

$$(|\theta'|^{m-1}\theta')'(\eta) - \frac{(p-m)}{p(m+1)-2m}\eta\theta'(\eta) - \frac{m}{p(m+1)-2m}\theta(\eta) \geq 0 \tag{4.2}$$

for $\eta \in \{\eta > 0 \,|\, \theta(\eta) > 0\}$ and

$$-|\theta'(0)|^{m-1}\theta'(0) \leq \theta^p(0). \tag{4.3}$$

The system of inequalities (4.2), (4.3) admits the following compactly supported solution:

$$\theta(\eta) = A(a-\eta)_+^{\frac{m}{m-1}}, \tag{4.4}$$

where the positive constants A, a are such that

$$\left(\frac{m}{m-1}\right)^m A^{m-1} \geq \frac{ma}{p(m+1)-2m}, \tag{4.5.1}$$

$$a^{\frac{(p-1)m}{m-1}} A^{p-m} \geq \left(\frac{m}{m-1}\right)^m. \tag{4.5.2}$$

The first of these insures that (4.2) holds while the second insures that (4.3) holds. As in subsection 3.1 one can easily check that such A, a exist for any $p > p_0$.

The regularity of $\theta(\eta)$ in (4.4) is sufficient for $\underline{u}$ to be a weak subsolution of problem (G_m). See [K]. (Indeed, we do not need the requirement that the subsolution have a continuous derivative $\underline{u}_x$ on the corresponding interface.) Hence, by comparison we deduce that the solution of (G_m) blows up in a finite time provided there exists T_0 such that

$$u_0(x) \geq \underline{u}(x,0) \equiv T^{-\frac{m}{p(m+1)-2m}} A \left(a - xT^{-\frac{p-m}{p(m+1)-2m}}\right)_+^{\frac{m}{m-1}}, \tag{4.6}$$

since then $u(x,t) \geq \underline{u}(x,t)$. It is clear that if $u_0(\cdot)$ is large enough, $T >> 1$ and $a << 1$ can always be found so that (4.5) and (4.6) hold. The blow up time for the corresponding solution of (G_m) cannot exceed T.

4.2 *Any* $u \not\equiv 0$ *blows up if* $p_0 < p < 2m$.

Problem (G_m) possesses self-similar solutions in $\mathbb{R} \times \mathbb{R}_+$ with $u_x(0,t) = 0$ of the form

$$\begin{aligned} u(x,t) \equiv u_B(x,t) &= (\tau+t)^{-\frac{1}{2m}} g(\xi), \\ \xi &= \frac{x}{(\tau+t)^{\frac{1}{2m}}}, \end{aligned} \tag{4.7}$$

where $g \geq 0$ solves the ordinary differential equation

$$(|g'|^{m-1}g')'(\xi) + \frac{1}{2m}\xi g'(\xi) + \frac{1}{2m}g(\xi) = 0 \tag{4.8}$$

and satisfies $g'(0) = 0, g(0) > 0$. The solution analogous to (3.10) is given by

$$\begin{aligned} g(\xi) &= D_m \left(d^{\frac{m+1}{m}} - \xi^{\frac{m+1}{m}}\right)_+^{\frac{m}{m-1}}, \\ D_m &= \left[\frac{1}{2m}\left(\frac{m-1}{m+1}\right)^m\right]^{\frac{1}{m-1}}. \end{aligned} \tag{4.9}$$

By employing the same comparison arguments as in Section 3 (inequalities (3.11)–(3.14)), we conclude that the solution $u(x,t)$ blows up in a finite time (which does not exceed $t_0 + t_* + T$) provided that there exist $\tau + t_*$ and T so large that

$$(\tau + t_*)^{-\frac{1}{2m}} >> T^{-\frac{m}{p(m+1)-2m}} \tag{4.10}$$

and

$$(\tau + t_*)^{-\frac{1}{2m}} << T^{-\frac{p-m}{p(m+1)-2m}}. \tag{4.11}$$

Clearly, for fixed τ, t_* both (4.10), (4.11) hold for $T >> 1$ if

$$\frac{p-m}{p(m+1)-2m} < \frac{m}{p(m+1)-2m}, \tag{4.12}$$

that is, if $p_0 = \frac{2m}{m+1} < p < 2m$. From this we conclude that if $p \in (p_0, 2m)$, every nontrivial solution is nonglobal.

4.3 *Global small solutions for $p > 2m$.*

We next construct a global weak supersolution of the (self-similar) form (cf. (4.1))

$$\begin{aligned} \overline{u}(x,t) &= (T+t)^{-\frac{m}{p(m+1)-2m}} f(\eta), \\ \eta &= \frac{x}{(T+t)^{\frac{p-m}{p(m+1)-2m}}}, \end{aligned} \tag{4.13}$$

where $f \geq 0$ satisfies

$$(|f'|^{m-1}f')'(\eta) + \frac{(p-m)}{p(m+1)-2m}\eta f'(\eta) + \frac{m}{p(m+1)-2m} f(\eta) \leq 0 \tag{4.14}$$

for $\eta \in \{\eta > 0 \,|\, f(\eta) > 0\}$ and

$$-|f'(0)|^{m-1}f'(0) \geq f^p(0). \tag{4.15}$$

Upon substitution of $f(\eta) = Bg(\eta + b)$ with $b \in (0, d)$ into (4.14) (4.15), and noting from (4.8) that

$$(|g'|^{m-1}g')'(\eta + b) = -\frac{1}{2m}(\eta + b)g'(\eta + b) - \frac{1}{2m}g(\eta + b)$$

we obtain (cf. (3.24), (3.26))

$$\begin{aligned} &-g'(\eta+b)\left[\frac{(\eta+b)B^{m-1}}{2m} - \frac{(p-m)\eta}{p(m+1)-2m}\right] \\ &+g(\eta+b)\left[\frac{m}{p(m+1)-2m} - \frac{B^{m-1}}{2m}\right] \leq 0 \end{aligned} \tag{4.16}$$

for $\eta \in (0, d-b)$ and

$$(BD_m)^{p-m}\left(d^{\frac{m+1}{m}} - b^{\frac{m+1}{m}}\right)^{\frac{m(p-1)}{m-1}} \leq \left(\frac{m+1}{m-1}\right)^m b. \tag{4.17}$$

Substitution of the formula for $g(\eta + b)$ given in (4.9) into (4.16) leads to the equivalent inequality (with $z \equiv \eta + b \in (b, d)$)

$$\begin{aligned} &-\left(\frac{1-B^{m-1}}{m-1}\right) z^{\frac{m+1}{m}} + \left(\frac{(p-m)(m+1)}{(m-1)(p(m+1)-2m)}\right) bz^{\frac{1}{m}} \\ &\qquad - \left(\frac{B^{m-1}}{2m} - \frac{m}{p(m+1)-2m}\right) d^{\frac{m+1}{m}} \leq 0. \end{aligned} \tag{4.18}$$

Since $p > 2m$ we may choose $B > 0$ such that

$$\frac{2m^2}{p(m+1)-2m} < B^{m-1} < 1. \tag{4.19}$$

Now set

$$\begin{aligned} e_1 &\equiv \frac{(1-B^{m-1})}{m-1} > 0, \\ e_2 &\equiv \frac{m+1}{m-1}\frac{p-m}{p(m+1)-2m} > 0, \\ e_3 &\equiv \frac{B^{m-1}}{2m} - \frac{m}{p(m+1)-2m} > 0, \end{aligned}$$

and observe that under these circumstances (with $d = \alpha b$ and $\alpha > 1$) (4.18) holds if and only if

$$R(z) \equiv -e_1 z^{\frac{m+1}{m}} + e_2 b z^{\frac{1}{m}} - (\alpha b)^{\frac{m+1}{m}} e_3 \leq 0. \tag{4.20}$$

Now it is clear that $R(z)$ is concave function of $z^{\frac{1}{m}}$ which takes its maximum at

$$z_* = \frac{be_2}{(1+m)e_1}.$$

Consequently $R(z) \leq R(z_*) \leq 0$ provided

$$\alpha^{\frac{m+1}{m}} \geq \frac{m}{m+1}\frac{e_2}{e_3}\left[\frac{e_2}{(m+1)e_1}\right]^{\frac{1}{m}} \equiv \overline{\alpha}^{\frac{m+1}{m}}. \tag{4.21}$$

We see that (4.17) will hold for $\alpha > \max(1, \overline{\alpha})$, provided b is chosen so small that

$$(BD_m)^{p-m}\left(\alpha^{\frac{m+1}{m}} - 1\right)^{\frac{m(p-1)}{m-1}} b^{\frac{p(m+1)-2m}{m-1}} \leq \left(\frac{m+1}{m-1}\right)^m. \tag{4.22}$$

Thus we have established the existence of a nontrivial global supersolution and hence completed the proof of Theorem G.

Remark 4.1: If $p = p_0 = 2m/(m+1)$, then the problem (G_m) admits the global self-similar solution (3.28) where $\beta = \alpha(m-1)/(m+1)$ and $h(\xi) \geq 0$ solves the problem

$$(4.23) \qquad \left(|h'|^{m-1}h'\right)'(\xi) + \beta\xi h'(\xi) - \alpha h(\xi) = 0 \quad \text{for } \xi > 0,$$

$$(4.24) \qquad -|h'(0)|^{m-1}h'(0) = h^{p_0}(0).$$

Equation (4.23) may be easily transformed into a first order equation. A simple analysis yields the existence and uniqueness of a nonincreasing compactly supported solution $h \not\equiv 0$. By the usual comparison arguments this implies the existence of global solution of problem (G_m) if $p \in [1, p_0]$. Notice that in this case as well a similar comparison argument can be applied by using a supersolution of self-similar form instead of the explicit solution.

Remark 4.2: Notice that we may reduce the equation (G_m) to the equation (P_m). To see this, assume for simplicity that $u_0 \in C^1$, is nonincreasing and has compact support. Then by the Maximum Principle ([K]), it follows that $u_x(x,t) \leq 0$ in $\mathbb{R}_+ \times (0,T)$. Defining $v(x,t)$ through

$$(4.25) \qquad u(x,t) = \int_x^\infty v(\xi,t)\,d\xi,$$

we deduce that $v = -u_x \geq 0$ solves the initial boundary value problem

$$(4.26) \qquad v_t = (v^m)_{xx} \qquad \text{in } \mathbb{R}_+ \times (0,T),$$

$$(4.27) \qquad v(x,0) = -u_0'(x) \qquad \text{in } \mathbb{R}_+,$$

$$(4.28) \qquad v^m(0,t) = \left(\int_0^\infty v(x,t)\,dx\right)^p \qquad \text{for } t \in (0,T).$$

Notice that the boundary condition (4.28) is *nonlocal.* Thus we arrive at the fact that $p_c = 2m$ is the critical exponent for this problem. This result could also be proved directly for problem (4.26)–(4.28) by using the comparison arguments with sub- and supersolutions. These comparison principles can be applied to this nonlocal problem since the problem generates a parabolic operator having the correct monotonicity properties. However, the long-term goal of our research is to extend the problems considered here to several space dimensions. In such cases, simple reductions as given here are probably not possible and it is therefore better to treat the problems separately.

4.4 *Any $u \not\equiv 0$ blows up if $p = 2m$.*

The proof is similar to that given in subsection 3.4 with g as in (3.31) given by (4.9) and the rescaled function (cf. (3.32))

$$\theta(\zeta,\tau) = (1+t)^{\frac{1}{2m}} u(\zeta(1+t)^{\frac{1}{2m}}, t)$$

which is related to the self-similar solution (4.7). We omit the details.

References

[BGK] F. V. Bunkin, V. A. Galaktionov, N. A. Kirichenko, S. P. Kurdyumov and A. A. Samarskii, *Localization in an ignition problem*, Doklady Akademii Nauk SSSR, Ser. Math. **302** (1988), 68–71.

[BL] C. Bandle and H. A. Levine, *On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains*, Transactions of the American Mathematical Society **655** (1989), 494–624.

[F] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, N.J., 1964.

[Fi] M. Fila, *Boundedness of global solutions for the heat equation with nonlinear boundary conditions*, Commentationes Mathematicae Universitatis Carolinae **30** (1989), 479–484.

[FQ] M. Fila and P. Quittner, *The blow-up rate for the heat equation with a non-linear boundary conditions*, Mathematical Methods in the Applied Sciences **14** (1991), 197–205.

[G1] V. A. Galaktionov, *On global nonexistence and localization of solutions to the Cauchy problem for some class of nonlinear parabolic equations*, Z. Vychisl. Matem. i Matem. Fiz. **23** (1983), 1341–1354; English transl.: USSR Comput. Math. and Math. Phys **23** (1983), 36–44.

[G2] V. A. Galaktionov, *Blow-up for quasilinear heat equations with critical Fujita's exponents*, Proceedings of the Royal Society of Edinburgh **124A** (1994), 517–525.

[GKMS] V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov and A. A. Samarskii, *On blowing-up solutions to the Cauchy problem for the parabolic equation* $u_t = \nabla(u^\sigma \nabla u) + u^\beta$, Doklady Akademii Nauk SSSR. Ser. Math. Phys. **252** (1980), 1362–1364; English transl.: Soviet Physics Doklady **25** (1980), 458–459.

[GKS] V. A. Galaktionov, S. P. Kurdyumov and A. A. Samarskii, *On the method of stationary states for quasilinear parabolic equations,* Matematicheskii Sbornik **180** (1989), 995–1016; English transl.: Mathematics of the USSR-Sbornik **67** (1990), 449–471.

[GP] B. H. Gilding and L. A. Peletier, *On a class of similarity solutions of the porous media equation,* Journal of Mathematical Analysis and Applications **55** (1976), 351–364.

[K] A. S. Kalashnikov, *Some problems of the qualitative theory of nonlinear degenerate parabolic equations of second order,* Uspekhi Matematicheskikh Nauk **42** (1987), 135–176; English transl.: Russian Mathematical Surveys **42** (1987), 169–222.

[L1] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + \mathcal{F}(u)$, Archive for Rational Mechanics and Analysis **51** (1973), 371–386.

[L2] H. A. Levine, *The role of critical exponents in blow up theorems,* SIAM Review **32** (1990), 262–288.

[LaP] M. Langlais and D. Phillips, *Stabilization of solutions of nonlinear degenerate evolution equations,* Nonlinear Analysis. Theory, Methods & Applications **9** (1985), 321–333.

[LP] H. A. Levine and L. E. Payne, *Nonexistence theorems for the heat equation with non-linear boundary conditions and for the porous medium equation backward in time,* Journal of Differential Equations **16** (1974), 319–334.

[SGKM] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations,* Nauka, Moscow, 1987 (in Russian; English transl: Walter de Gruyter, Berlin, 1995).

[S] D. H. Sattinger, *Topics in Stability and Bifurcation Theory,* Lecture Notes in Mathematics **309**, Springer-Verlag, New York, 1973.